On the hologram of de Sitter

Bruno Carneiro da Cunha*

Enrico Fermi Institute and Department of Physics, The University of Chicago, 5640 South Ellis Avenue, Chicago, Illinois 60637

A careful reduction of three-dimensional gravity to the Liouville description is performed, where all gauge fixing and on-shell conditions come from the definition of asymptotic de Sitter spaces. The roles of both past and future infinities are discussed and the conditions space-time evolution imposes on both Liouville fields are made explicit. Space-times which correspond to nonequivalent profiles of the Liouville field at $\mathcal{I}^-$ and $\mathcal{I}^+$ are shown to exist. The qualitative implications of this for any tentative dual theory are presented.

I. INTRODUCTION

The importance of understanding quantum gravity in de Sitter space can hardly be overstressed. Touching on physical problems such as the dynamics of inflation and the eventual fate of the universe, the question is in itself a necessary step toward a background-free formulation of the quantum theory of gravity and quantum cosmology.

Not having a precise string theory background in which to embed the space, our understanding relies on extrapolations of ideas gathered in the study of black holes and anti–de Sitter (AdS) spaces. A particularly useful idea is holography [1], which found a realization for AdS backgrounds [2]. With this in mind, Strominger gave a prescription [3] to recover data pertaining to the space-time based on a hypothetical dual (holographic) theory. The arguments in support of this prescription are numerous [4,5], although few seem to tackle properties of dS space which do not parallel those of AdS space.

The properties of dS space which make the formulation of holography particularly difficult are the existence of a horizon and the compactness of spatial slices. The former poses problems for unitary evolutions of Hilbert spaces assigned to Cauchy surfaces and the latter makes unclear the procedure to define conserved charges. Horizons are in fact already known to cause problems in AdS space, where the formulation of (perturbative) string theory in, say, Bañados-Teitelboim-Zanelli (BTZ) backgrounds is an open problem.

Also, if one chooses to think about holography in a covariant way [6], the boundary theory seems to live in two disjoint codimension 1 surfaces, the past and future infinities, $\mathcal{I}^-$ and $\mathcal{I}^+$, respectively. Strominger gave some arguments in favor of a reduction which would map states from one to the other. Those were based on the initial value formulation which would determine the final state. Whether this can be brought forward to the full quantum theory is unclear.

The aim of this article is to give an alternative way of regarding this map, at least for three-dimensional space-times. The arguments presented here imply that the mapping is a truncation. One can argue that the full theory does live in both infinities, but some states (which we will call static) on it corresponding to singularity-free solutions can be "projected" to either infinity by a gauge choice. We will also consider instances where such a projection fails.

The paper is organized as follows. In Sec. II a reduction from the Chern-Simons formulation to Liouville theory is carried out. We choose a different "gauge fixing" from previous work [7] in order to keep the space-time picture explicit. In Sec. III we are able to give a simple interpretation for the Liouville field, and then state the generic conditions under which an asymptotically de Sitter space-time will be described by it. In Sec. IV we discuss the Strominger mapping and problems arising from cosmological singularities. We then conclude with a prospectus of what else classical de Sitter gravity could say about holography.

II. CHERN-SIMONS AND WESS-ZUMINO FORMULATIONS REVISITED

Three-dimensional gravity has been the subject of extensive research over the past two decades [8]. The Chern-Simons formulation [9,10] makes explict the role of local isometry invariance and also the fact that the theory is locally trivial. Sticking to the positive cosmological constant case, we can write the action in terms of an SL(2,C) connection, defined as $A_a^i = \omega_a^i + i\ell^{-1} e_a^i$, where e_a^i is the dreibein and ω_a^i is the Lorentz dual to the spin connection. Then the properties above are readily seen by direct inspection of the action,

$$S = \frac{\ell}{4\pi G} \mathfrak{I} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$
$$+ B(A, \bar{A}) \tag{1}$$

where $A = A^i \gamma_i$, $\bar{A}$ is the complex conjugate to A, and $\mathfrak{I}$ denotes the imaginary part. γ_i can be though of as a representation of the SL(2,C) algebra in three dimensions, satisfying $2\,\mathrm{Tr}[\gamma_i \gamma_j] = \eta_{ij}$ and $2\,\mathrm{Tr}[\gamma_i \gamma_j \gamma_k] = \epsilon_{ijk}$. $B(A, \bar{A})$ is a boundary term required to continue interior solutions to the causal boundary of space-time. The properties of isometry invariance and local triviality translate, respectively, to trivial facts about Chern-Simons forms (i) they are gauge invariant and (ii) their variation with respect to A is a closed form. The dynamics is then determined by the boundary data $B(A, \bar{A})$, namely, which particular space the dynamic space-time ap-

*Electronic address: bcunha@theory.uchicago.edu

proaches asymptotically. In the case at hand the sensible choice is to have the metric approach the de Sitter metric at past and future infinity.

In the spirit of classical general relativity, one should strive to give a coordinate independent formulation of an asymptotic de Sitter space. The main advantage of this approach is to make a clear distinction between what is considered "gauge choice" in the Chern-Simons language and what is considered "on shell." Following Shiromizu *et al.* [11] (after similar work done for AdS space by Hawking and by Ashtekhar and Magnon [12]), we will say that an *n*-dimensional space-time with metric g_{ab} is *asymptotically de Sitter* when the following conditions hold.

(1) *The causal boundary of space-time is* $\mathcal{I}^+ \cup \mathcal{I}^-$ [13]. Specifically, there are two functions Ω^- and Ω^+ which vanish at past and future infinities, respectively, but not their derivatives; also, the corresponding unphysical or fiducial metric $\hat{g}_{ab}^{\pm} \equiv (\Omega^{\pm})^2 g_{ab}$ is regular there.

(2) *The geometry is asymptotically trivial*: the space-time (M,g) is a solution of the equations $R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$, where $(\Omega^{\pm})^{-n+1} T_b^a$ has a smooth limit to the boundary.

The first item above tells us that Ω is a nonsingular coordinate near either $\mathcal{I}^-$ or $\mathcal{I}^+$. Furthermore, its gradient is normal to the boundary and hence timelike. So one is naturally led to the *gauge choice* where the time function is given by $t \sim \log \Omega^-$ near $\mathcal{I}^-$. Whether or not one can stick to this choice up to $\mathcal{I}^+$ depends on global issues like the causal structure of space-time and whether the function t thus defined can be extended globally. We will address these points in Sec. III. Assuming that this is a good gauge choice throughout the evolution of space-time amounts to assuming that the spatial slices will not undergo topology change. In fact, the assumption already furnishes us with a Morse function, if we see time evolution as a cobordism. There is more to say about the instances when such gauge fixing breaks down, but we will postpone it for now.

The second item in the list relates to solving the equations of motion near past and future infinity. Here we will be mostly interested in the case of pure gravity, so $T_{ab} = 0$ and the metric is locally the de Sitter vacuum. It is interesting to note that if one takes holography as a property of quantum gravity, the latter will be represented in de Sitter backgrounds as a transfer matrix interpolating between asymptotic de Sitter spaces at past and future infinity. The requirement that these spaces satisfy item 2 above echoes the Lehmann-Symanzik-Zimmermann (LSZ) reduction between a general Green's function of any quantum field theory and its S matrix elements. Since the latter are defined for on-shell elements, it is unclear at this point how the quantum fluctuations of the bulk fields would be encoded in the dual theory. As it stands, even in the AdS conformal field theory (CFT) these are encoded in a complicated way in the boundary theory. Only gauge-invariant observables have a clear-cut correspondence in terms of bulk fields.

In the following we will apply the conditions above to the metric at $\mathcal{I}^-$ (and as such we will drop the superscript). In

terms of the unphysical or fiducial variables, the Einstein equations in *n* dimensions read

$$\hat{R}_{ac} + (n-2)\frac{1}{\Omega}\hat{\nabla}_a \hat{\nabla}_c \Omega + \hat{g}_{ac}\hat{g}^{de}\frac{1}{\Omega}\hat{\nabla}_d \hat{\nabla}_e \Omega$$

$$-(n-1)\hat{g}_{ac}\hat{g}^{de}\frac{1}{\Omega^2}\hat{\nabla}_d \Omega \hat{\nabla}_e \Omega = \frac{2\Lambda}{n-2}\frac{1}{\Omega^2}\hat{g}_{ac}. \quad (2)$$

Since the unphysical Ricci tensor is regular, the terms diverging with some power of Ω must vanish at $\partial \hat{M}$. The term proportional to Ω^{-2} yields

$$\hat{g}^{ab}\hat{\nabla}_a \Omega \hat{\nabla}_b \Omega = -\frac{2}{(n-2)(n-1)}\Lambda \text{ at } \mathcal{I}^-. \quad (3)$$

Back in three dimensions, (3) means that one can define a timelike vector field $\xi^a \equiv \ell g^{ab}\nabla_b \log \Omega$, with $\Lambda = \ell^{-2}$, which becomes normalized in the physical space as one approaches $\mathcal{I}^-$. The integral curves of this vector are parametrized by a time function, which we will call $t \equiv \ell \log \Omega$. One can actually use part of the gauge symmetry of Ω and eliminate the order Ω corrections to $\hat{\xi}^a$ [14]. With this provision, the subleading term implies

$$\hat{\nabla}_a \hat{\nabla}_c \Omega + \hat{g}_{ac}\hat{g}^{de}\hat{\nabla}_d \hat{\nabla}_e \Omega = 0 \text{ at } \mathcal{I}^- \quad (4)$$

which in turn means that $\hat{\nabla}_a \hat{\nabla}_c \Omega = 0$. On top of those, there is a remnant gauge condition, which allows it to be multiplied by a generic nonvanishing function of the transverse coordinates. Since the spatial slices are topologically spheres, one can use up the reparametrization invariance and write the spatial metric as conformally flat. The conformal factor can then be absorbed by a redefinition of t. The fiducial metric can then be considered flat.

It is a straighforward exercise to compute the physical dreibein and spin connection given their fiducial counterparts and the function t:

$$e_a^i = e^{-\ell^{-1}t}\hat{e}_a^i, \quad (5)$$

$$\omega_a^i = \hat{\omega}_a^k - \frac{1}{\ell}\epsilon_{ij}^k e_t^i e_a^j, \quad (6)$$

where $e_t^i = \xi^a e_a^i$. Using local Lorentz invariance, one can write $e_t^i = \delta_0^i$ at $\mathcal{I}^-$. Also, given that the fiducial metric is flat at $\mathcal{I}^-$, $\hat{\omega}_a^k = 0$ there, and as a result the corresponding component of the gauge field satisfies

$$\left. \begin{array}{l} A_a^+ \equiv A_a^1 + iA_a^2 = 0 \\[2mm] A_z^0 = A_{\bar{z}}^0 = 0 \end{array} \right\} \text{ at } \mathcal{I}^-, \quad (7)$$

so we can choose

$$A_t^0 = \frac{i}{\ell} \quad (8)$$

as a gauge fixing for Eq. (1). As said before, we will see Eq. (7) as the on-shell condition, to be imposed at $\mathcal{I}^-$, and Eq. (8) as a gauge choice.

The variation of the the action (1) gives [15,16]

$$\delta S = \frac{k}{8\pi}\Im\left[\int_{\partial M}\mathrm{Tr}(\delta A\wedge A)+2\int_M\mathrm{Tr}(\delta A\wedge F)\right]$$
$$+\delta B \qquad (9)$$

where $F=dA+A\wedge A$ and the ''level'' of the model is $k=\ell/2G$. One sees that the stationary points of the action correspond to flat connections. The allowed solutions are those which can be continued to the boundary, i.e., whose boundary variation vanishes. This of course depends on the exact form of B. With our conditions (7) and (8) above, we can see that the boundary term in Eq. (9),

$$\mathrm{Tr}(\delta A\wedge A)=-\frac{1}{2}\delta A^0\wedge A^0+\delta A^-\wedge A^++\delta A^+\wedge A^-,\qquad (10)$$

vanishes identically for connections giving rise to asymptotically de Sitter spaces. At $\mathcal{I}^-$ we can see it directly from the conditions (7), (8). First, one sees that A^+ is fixed at zero, and then the corresponding term vanishes, along with the one containing δA^+. Furthermore, A^0 is orthogonal to the boundary, by Eq. (7). At $\mathcal{I}^+$ one can parallel the argument to find that the conditions (7) hold with A^- instead of A^+. This comes about because we need to define the time function as $t=-\ell\log\Omega^+$ in order to have $\mathcal{I}^+$ at $t=\infty$. Then, by setting $B=0$, we make sure that only connections which give rise to asymptotically de Sitter spaces are singled out.

Considering the on-shell and gauge fixing conditions, one may ask what is the resulting action after these are enforced in Eq. (1). The discussion above implies that A_t^i is not a dynamical variable but rather a Lagrange multiplier. In fact, decomposing $d=dt\partial/\partial t+\tilde{d}$ and $A=A_t+\tilde{A}$, we can write the action as

$$S=\frac{k}{8\pi}\Im\left\{\int_M\mathrm{Tr}(\tilde{A}\wedge\dot{\tilde{A}}\wedge dt)+2\int_M\mathrm{Tr}[A_t\wedge\tilde{F}]\right.$$
$$\left.-\int_M\mathrm{Tr}[\tilde{d}(\tilde{A}\wedge A_t)]\right\}; \qquad (11)$$

the last term vanishes due to the choice of coordinates. Now, seeing A_t as a Lagrange multiplier, we can integrate over it and then enforce the constraint that $\tilde{F}=\tilde{d}\tilde{A}+\tilde{A}\wedge\tilde{A}=0$. For simply connected manifolds, the solution of the constraint is $\tilde{A}=-\tilde{d}\mathcal{G}\mathcal{G}^{-1}$ with $\mathcal{G}$ an SL(2,C)-valued function of the manifold. If the spatial slices have punctures, we will have to deal with holonomies of $\mathcal{G}$. We will postpone this discussion to the next section. The on-shell condition now translates into $A_t=-\dot{\mathcal{G}}\mathcal{G}^{-1}=i\ell^{-1}\gamma_0$, meaning that we can write $\mathcal{G}=e^{-i\ell^{-1}t\gamma_0}g(z,\bar{z})$, with g independent of t.

Changing variables in Eq. (11), and using the fact that the resulting action depends only on the boundary values of $\mathcal{G}$, one arrives at

$$S[g]=-\frac{k}{4\pi}\Im\left\{\int\mathrm{Tr}(g^{-1}\partial g g^{-1}\bar{\partial}g)d^2x\right.$$
$$\left.+\frac{i}{6}\int_M\mathrm{Tr}[(g^{-1}dg)^{\wedge 3}]\right\} \qquad (12)$$

where we used $d^2x=(i/2)dz\wedge d\bar{z}$ as the measure of the boundary. One notes that g actually gives information about the spatial slices of the metric, or the fiducial metric, since with it one can recover the physical metric at $\mathcal{I}^-$ by the conditions (7) and (8). It is also clear by these conditions that not all g will give rise to an acceptable metric. First and foremost, the metric on the sphere has to be conformally flat, that is to say, we have already made a choice for the Kähler structure on the spatial slices. However, by inspecting the form of A near $\mathcal{I}^-$,

$$A=\frac{i}{\ell}dt\,\gamma_0-e^{-i\ell^{-1}t\gamma_0}\tilde{\partial}g g^{-1}e^{i\ell^{-1}t\gamma_0}, \qquad (13)$$

we see that A^+ vanishes at $t=-\infty$ independently of g and then it only makes sense to impose the flat condition through A^-. By implementing the first-class constraint on Eq. (12) via the usual Lagrange multiplier, one arrives at the improved action

$$S_I[g,\Theta]=S[g]+\frac{k}{2\pi}\Im\int d^2x\,\mathrm{Tr}\left[\Theta\left(\partial g g^{-1}-\frac{i}{\ell}\gamma_+\right)\right] \qquad (14)$$

where $\Theta=\theta\gamma_-$ and θ is a scalar function. It is well known [17] that the action above has a gauge symmetry

$$g\rightarrow h(z,\bar{z})g, \quad \Theta\rightarrow h\Theta h^{-1}-dhh^{-1} \qquad (15)$$

for h in the lower triangular subgroup of SL(2,C). The invariance when h is a function of $\bar{z}$ stems from the usual left-right symmetry of Eq. (12). Parametrizing g by the Gauss product

$$g=e^{\chi\gamma_-}e^{2i\rho\gamma_0}e^{\varphi\gamma_+} \qquad (16)$$

we can write Eq. (14) as

$$S_I[g,\theta]=-\frac{k}{2\pi}\Im\int d^2x\left[\partial\rho\bar{\partial}\rho+e^{2\rho}\partial\varphi\bar{\partial}\chi\right.$$
$$\left.+\theta\left(e^{2\rho}\partial\varphi-\frac{i}{\ell}\right)\right]. \qquad (17)$$

The solution of the equation of motion for θ is then

$$e^{2\rho}=\frac{i}{\ell}(\partial\varphi)^{-1} \qquad (18)$$

and one can use the gauge freedom (15) to set

$$\chi = -\frac{i\ell}{2}\frac{\partial^2 \varphi}{\partial \varphi}. \tag{19}$$

We will see later why this is a natural choice. The resulting action is then

$$S_I = \frac{k}{8\pi}\Im\int d^2x\left[\frac{\partial^2\varphi\,\partial\bar{\partial}\varphi}{(\partial\varphi)^2} - 2\frac{\partial^2\bar{\partial}\varphi}{\partial\varphi}\right], \tag{20}$$

which can be seen to be equivalent to the Liouville action if we write its stress tensor, using the Nöther procedure,

$$T = \frac{k}{4\pi}\left[\frac{\partial^3\varphi}{\partial\varphi} - \frac{3}{2}\left(\frac{\partial^2\varphi}{\partial\varphi}\right)^2\right] = \frac{k}{4\pi}\{\varphi;z\}, \tag{21}$$

or, in other words, the (unrestricted) stress tensor encodes the Virasoro currents' Poisson brackets, which is a universal characteristic of the Liouville field. In the equation above $\{\varphi;z\}$ denotes the Schwarzian derivative. This hardly comes as a surprise since Eq. (20) is reminiscent of the natural geometrical action for a particular coadjoint orbit of the Virasoro group (when the symmetry group is real and compact and $b_0 = 0$ in the notation of [18]). The fact that the orbit in that case is equivalent to a highest weight representation has led to many conflicting arguments about the entropy of de Sitter spaces. We will have nothing further to say about this thorny question here.

III. THE BROWN-HENNEAUX "SYMMETRY" AND THE MAPPING

A useful way to think about the action (12) is to consider two separate actions, depending on g and $\bar{g}$, and to impose the reality condition on the equations of motion. The solutions of Eq. (12) for g can then be seen as holomorphic currents. The action of $SL(2,C)$ which brings one solution of Eq. (17) into another is

$$g \to e^{i\,\log(\partial w)\gamma_0}e^{(i\ell/2)\partial\,\log(\partial w)\gamma_-}g,$$

$$z \to w(z). \tag{22}$$

One can see directly from Eq. (18) that this maintains the Kähler structure of the spatial slices at $\mathcal{I}^-$. Since it amounts to a would-be "pure gauge" transformation, it can be realized by a coordinate transformation, which involves redefining the time function t to arrive at another metric which asymptotes to the de Sitter metric at $\mathcal{I}^-$ [19]. The current coming from the gauge-fixed g is

$$J_L = -\partial g\,g^{-1}dz = \frac{i}{\ell}\gamma_+dz + \frac{i}{2k\ell}L(z)\gamma_-dz. \tag{23}$$

So the action (22) is to transform $L(z)$ into

$$\tilde{L}(w) = \left(\frac{\partial w}{\partial z}\right)^{-2}\left[L(z) - \frac{k}{4}\{w;z\}\right]. \tag{24}$$

We recognize above the anomalous transformation law of the stress tensor, with central charge $3k = 3\ell/2G$. One then sees

the reason for picking Eq. (19): it cancels the factor of the current proportional to γ_0 and thus maintains the former on-shell condition (7).

It is also transparent from the previous discussion that, if one is able to stick to the gauge choice (8) throughout time evolution, the single Liouville mode found above will give rise to a globally defined metric. For $L(z)$ defined on a sphere with two single poles, these solutions have been discussed before [19,25], and have been found to be in the Kerr–de Sitter class. The Brown-Henneaux transformation discussed above generates all such solutions for meromorphic coordinate transformations. Even more generically, one can create higher genus space slices in this manner.

However, the effect of the Liouville field in the metric is not felt until one leaves $\mathcal{I}^-$, and in the bulk of space-time any metric is locally gauge equivalent to the trivial metric. Then, if one is to assign a gauge-invariant meaning to the Liouville field, one is forced to see a profile of it as just some parametrization of the conserved charges of the reduced system, which includes the holonomies of the gauge field as its simple poles. Echoing the study of Riemann surfaces, it is the uniformizing coordinate which encodes the properties of the spatial slices. Being a coordinate choice, it only fails to be a well-defined gauge transformation because it lives naturally at the boundary.

If we consider as an example the Kerr–de Sitter class, one could have performed the same reduction as in last section for $\mathcal{I}^+$. One would then find another Liouville profile there, which would encode the same conserved charges. In particular, the positions of the simple poles for both Liouville modes would be related by a global $SL(2,C)$ transformation. As such the Liouville profiles are essentially equivalent, amounting just to a global frame in which the global quantities—like mass and angular momentum—are measured. This is realized in the discussion of the last section by the fact that, if one is able to fix the gauge (8) throughout the evolution of space-time, the gauge connection, and consequently the metric, would be globally written as $A = -d\mathcal{G}\mathcal{G}^{-1}$, with $\mathcal{G} = e^{-i\ell^{-1}t}g(z,\bar{z})$, and then the reduced action (12) would be the same for both pieces of the boundary.

For generic configurations, the requisition that the gauge is fixed once and for all seems excessive. Stable causality implies only that there is a globally well-defined time function, in fact it is equivalent to it, but it does not imply that this time function is such that its vector field is geodesic and everywhere normalized, which are the hidden assumptions behind Eqs. (7) and (8). Topologically, one can see time evolution as a cobordism, and this time function as a Morse function. The mere existence of this Morse function then implies that the degrees and number of poles in the Liouville field, as well as their residues, have to be conserved during time evolution. When this happens, we will say that the Liouville profiles at $\mathcal{I}^-$ and $\mathcal{I}^+$ are in the same homotopic class. This is how the intuition of separated worldlines of point particles is imprinted in the Liouville fields. Generically, lifting this condition would give rise to time evolutions where the particles could join or split as time progresses, although one would like at least to enforce conservation of total mass and angular momentum to talk about reasonable

spaces. In fact, in this case the condition to enforce on the space-time seems to be strong causality, for splittings or joinings which involve small enough point masses would still respect the condition that there are no closed timelike curves. At any rate, the point here is that the relation between past and future Liouville profiles is tied to the time evolution of the system. Solutions which have been considered so far have the property that time evolution is geodesic and normalized. They correspond to equal profiles of the Liouville field at $\mathcal{I}^-$ and at $\mathcal{I}^+$.

One can get an insight into what are the solutions in which the Liouville profiles are not equivalent but are in the same homotopic class by considering boosted sources. "Static" solutions like the Kerr–de Sitter have the property that the matter current found via the equations of motion $F = *J$ is an sl(2,C)-valued one-form which is zero almost everywhere, but otherwise orthogonal to e_a^0. Since J is basically defined to be the Lorentz dual of the stress-energy tensor, this latter fact confirms our intuition of noninteracting point particles, or, in other words, a configuration of dust. However, if one makes a SL(2,C) transformation in the ith of those particles, $J_i \rightarrow U J_i U^{-1}$, the current may no longer be orthogonal to e_a^0, or even its transformed counterpart. This comes about because the triad depends on A and it complex conjugate, so its transformation will depend on U and its complex conjugate, whereas the matter current transforms by U alone.

Now consider the quantity $e^0 \wedge de^0$. By some manipulation one can relate it to $e^0 \wedge *J^0$:

$$
\begin{aligned}
e^0 \wedge de^0 &= \ell \, \mathfrak{I}(e^0 \wedge dA^0) \\
&= -\ell \, \mathfrak{I}(e^0 \wedge A^1 \wedge A^2) \\
&\quad + \ell \, \mathfrak{I}(e^0 \wedge *J^0) \\
&= -e^0 \wedge e^1 \wedge \omega^2 - e^0 \wedge \omega^2 \wedge e^1 \\
&\quad + \ell \, \mathfrak{I}(e^0 \wedge *J^0) \\
&= -e^0 \wedge de^0 + \ell e^0 \wedge \mathfrak{I}(*J^0),
\end{aligned}
$$

in which we used the definition of the connection in the first and third lines and the equations of motion in the second. The fourth line is the condition of zero torsion. So $e^0 \wedge de^0 = (\ell/2) \mathfrak{I}(e^0 \wedge *J^0)$ and thus it is zero when the matter current corresponds to a distribution of dust. As we discussed above, generically this will not be the case and this quantity will be nonzero.

The quantity $e^0 \wedge de^0$ measures the failure of the form e^0 to be hypersurface orthogonal. By Frobenius' theorem, this means, among other things, that the space-time does not allow a foliation in which the vector field $(e^0)^a$ is everywhere orthogonal to the hypersurfaces. The latter fact forbids us to choose the gauge (8) globally. So boosted source solutions will not be described by a single Liouville mode, but rather both field profiles, at $\mathcal{I}^-$ and $\mathcal{I}^+$, will be needed to reconstruct the space-time. Because asymptotically all matter behaves as dust, the gauge choice is appropriate for generic matter configurations only at $\mathcal{I}^-$ and $\mathcal{I}^+$. Asymptotically,

the description in terms of the Liouville fields found in the preceding section will be valid.

One notes that, if the dS/CFT correspondence is implemented in the same manner as in AdS spaces, the space-time information is recovered from the holographic theory via the renormalization group (RG) flows. These are first order in the scale parameter and hence are determined by the initial (UV) value of the couplings. By the de Sitter version of the UV/IR correspondence, this initial value would be naively set at either $\mathcal{I}^+$ or $\mathcal{I}^-$. The question that poses itself now is: how exactly is one supposed to tell the static from the boosted solutions found in the above discussion if a single Liouville field is blind to them? The aforementioned argument points to the fact that there are *two* sectors in the holographic theory, which are *independent* in the UV limit, but whose RG flow induces correlations. One could think of the two Liouville fields found above as parametrizing the (space-time) Virasoro current sector of the full quantum theory, its UV fixed point corresponding to both $\mathcal{I}^+$ and $\mathcal{I}^-$. The solutions considered so far, in which the gauge fixing (8) is valid globally, correspond to "diagonal" states which are the tensor products of equivalent states in each sector of the holographic theory. Although the particular field representation of the asymptotic symmetries will change with different dimensions, it seems plausible to expect that these generic considerations will still be valid.

IV. THE ROLE OF COSMOLOGICAL SINGULARITIES

In the last section we showed that there are space-time solutions corresponding to different states in the sectors $\mathcal{I}^-$ and $\mathcal{I}^+$. However, this is not how the known example of holography is implemented. In fact, in AdS/CFT the UV fixed point is obtained by a limiting process where degrees of freedom in the interior of the AdS space are integrated out. The limiting process effectively gives the asymptotic value of the fields near, say, $\mathcal{I}^-$ and then there is enough information to reconstruct the whole of space-time. This, of course, assumes that time evolution can indeed provide the form of the field in the bulk of space.

This assumption was hidden in the gauge choice (8) during the preceding discussion. As we saw in the preceding section, the assumption amounts to the constraints between the past and future profiles discussed in the last section, as dictated by stable causality. Extending what was discussed in the last section, let us consider where cases the spatial slices can undergo topology changing. We can have disassociation of punctures, as in a single puncture dividing into two or more, or even the formation of handles [20] and the generation of nonconnected components [21]. In the case of disassociation and joining of punctures, one is naturally led to associate them with Liouville profiles with different numbers of poles, but with the same total residue to preserve the conserved charges. As in flat space [22], there is a danger of clashing punctures creating closed timelike curves and thus violating strong causality. We will see that, also paralleling the flat space scenario [23], there is a mass bound for these types of processes over which a cosmological singularity

will be generated. One can then obtain the "big bang" scenario by a time reversal.

We can see how the classical process of cosmological collapse evolves by considering the initial value problem. The matter considered will be an arbitrary distribution of stationary dust $\rho(z,\bar{z})$. This can be seen as a distribution of punctures, arranged so that the total mass is finite. The calculation is similar to the one done for AdS space [24]. As we saw in the second section, the time evolution near $\mathcal{I}^-$ is given by a pure contraction:

$$\nabla_a \xi_b = -\frac{1}{\ell}(g_{ab} + \xi_a \xi_b) \text{ at } \mathcal{I}^-. \tag{25}$$

So the natural ansatz for the solution is

$$ds^2 = -dt^2 + [a(t)]^2 h_{ab} dx^a dx^b, \tag{26}$$

where h_{ab} is a Riemannian two-dimensional metric. In the language of the second section, it is the induced unphysical spatial metric. In these variables the equations of motion read

$$8\pi G a^2 \sqrt{h}\rho - \frac{1}{2}\sqrt{h}^{(2)}R = -\left(\frac{1}{\ell^2}a^2 - \dot{a}^2\right)\sqrt{h},$$

$$\ddot{a} - \frac{1}{\ell^2}a = 0. \tag{27}$$

The second equation means that the left hand side of the first equation is a constant of motion. Calling $\beta = \ell^{-2}a^2 - \dot{a}^2$ and integrating the first equation over the spatial coordinates, we find

$$\beta = \frac{\pi}{\Sigma}(\chi - 8GM) \tag{28}$$

where Σ is the area of the spatial slice in the unphysical metric and M is the total mass of the dust configuration. χ is the Euler-Poincaré characteristic of the spatial slices, which is 2 for the sphere. One sees that, when $4GM > 1$, β is negative. When that happens, a is given by

$$a(t) = \sinh\left(\frac{t-t_0}{\ell}\right) \tag{29}$$

with t_0 depending on M. Hence the "size of the universe" vanishes at some finite global time.

This shows that there are states in the boundary theory which correspond to space-times with cosmological singularities. It is interesting to note that this bound resembles the bound of masses which corresponds to operators in CFT with real scaling weight [3]. For fields whose mass is greater than the de Sitter mass, the natural Compton wavelength is smaller than the de Sitter length, so the dust approximation is expected to hold well at large times since then interactions and movements will be swamped by the cosmological constant. At small distances when the pressure begins to be non-negligible, one can resort to the study of the Raychaudhuri

equation to predict the appearance of the singularity. In fact, a simple calculation shows that, when the energy distribution of a perfect fluid whose equation of state is $P = w\rho$ satisfies

$$8\pi G\rho > \frac{1}{(1-w)\ell^2}, \tag{30}$$

the expansion parameter will diverge to $-\infty$ at finite proper time for $w < 1$. With all this in mind, one is then tempted to associate operators in CFT with complex (space-time) scaling weights to space-times with cosmological singularities [25]. Having no further description of the full quantum theory, one can only hope that the singularity is resolved there [26].

At any rate, the same considerations of the last section apply here: the past (and future) Liouville mode cannot see the difference between different space-time configurations, such as, for instance, those of a gas of moving particles and the dust configuration above. As in the case with punctures, details like relative velocities and local interactions are overwhelmed by the fact that distances between distinct points are growing exponentially with proper time. The question is then how this information is encoded in the boundary theory. As in the previous section, one is tempted to propose that the full Hilbert space of the theory is more than just the modes responsible for the Virasoro currents at $\mathcal{I}^-$. If the singularity is resolved, there may be a way to understand the final state as some "bounced" configuration at $\mathcal{I}^+$, if the space-time picture is recovered eventually.

V. WHITHER THE CORRESPONDENCE?

One of the big puzzles about holography is that it is deeply hidden in the classical theory, but it is manifest in the working quantum theories with gravitation that we have at the moment. The little we can infer about its properties is construed from extrapolations of ideas coming from the study of black holes and (A)dS (semi)classical dynamics, as in the preceding discussion. As limited as it is, general covariance allows us to make some statements about generalities of any prospective quantum theory.

In the present case, the most curious point is that the particular quantum theory which gives rise to a de Sitter background "lives" in disjoint regions of space-time. This fact comes about because the positions of the punctures at $\mathcal{I}^-$ and $\mathcal{I}^+$ are independent. In CFT dual language, by the state-operator correspondence, this means that the state at $\mathcal{I}^-$ may not be the same as that $\mathcal{I}^+$. In fact, as argued in the last section, one would generically like to specify those two independent states in the dual theory and then ask questions about space-time behavior. This also has some echoes of the study of quantum fields in an eternal black hole background [27] and more recently in global de Sitter spaces [4], where there is a distinction between the Schwarzschild Fock space and the Kruskal Fock space. Locality tells us that the former splits into two subsets, corresponding to the visible and not visible exterior regions of the extended space-time. The vacuum of the quantum field is a pure state in the Kruskal (or Hartle-Hawking) Fock space, but expectation values of local

operators in either exterior can only see the projection of the state into the corresponding Schwarzschild Fock space. Some arguments have been put forward to extend this point of view to gravity itself [28,29]. The results presented here point to the fact that this may be even more generic, in that regions separated by a horizon, be it of a black hole or of the observable universe, are described by a theory whose Fock space naturally decomposes into an enumerable set of subspaces (assuming that such a description is indeed valid at the UV fixed point). In this theory the existence of a horizon is encoded by correlations between the distinct subspaces, e.g., the $\mathcal{I}^{\pm}$ correlators found in [4], and locality outside the horizon is translated to the fact that observable quantities, the "true observables" of [4], are obtained by projection onto one particular subspace of the theory. Quantities which are defined in the whole of space-time are the metaobservables of [30].

In the AdS/CFT case these subtleties could be overlooked in simple deformations of the background since its holographic screen is connected, for in global AdS space there is no horizon. For dS/CFT, even in the simplest case, global de Sitter space, our naïveté is exposed. In the AdS case, even in the presence of a horizon one could consistently truncate the full spectrum of the theory into those states seen by the visible spatial infinity. The horizon is then seen as an impenetrable membrane [31]. For de Sitter space, this line of reasoning leads to a paradox [32]. The discussion in this paper implies that the kernel of the problem is in the truncation, which cannot be done consistently for generic perturbations of the global de Sitter background. Recently, this fact has been stressed by Bousso *et al.* [33], who tried to carry out this truncation for theories with interacting form fields. Although such fields are nonlocal excitations in the boundary CFT, and in this sense complementary to what has been done here, both results agree in casting doubt on the program of

setting up a quantum theory of gravity in de Sitter backgrounds with a finite number of degrees of freedom [34].

As is customary, there are more questions raised than answers. The particularly hard question is to understand those CFT states corresponding to $\beta < 0$ in Eq. (28). Or, in other words, how can we make sense of operators with complex scaling dimension in such a CFT. In dS_3 space one can create this type of background by a suitable identification of global de Sitter space with an element of the isometry group. In this case the identification is obviously singular but this particular type of singularity merited some attention recently [35]. Another, perhaps more amenable, question deals with the entropy of these backgrounds. As in the AdS case, the Liouville mode is the simple result of global covariance and therefore cannot distinguish between quantum states giving rise to the same geometry. This fact is encoded by semiclassical calculations showing that $c_{eff} = 1$ for the Liouville field [36]. On the other hand, it is extremely confusing how exactly the would-be gauge symmetries (the Brown-Henneaux type discussed in Sec. II) should act on the quantum states. It is unclear whether one can continue tackling these questions by thinking in terms of classical gravity alone.

When this manuscript was in its final phase of revision, the work of Balasubramanian *et al.* [37] appeared, in which there is some overlap with the points discussed in Sec. III and V.

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